Course Title: The Story of Numbers: From Zero to Infinity

Course Overview: This course is a survey of the origins and evolution of humanity's concept of number. It takes a global approach where the contributions of every civilization are considered, and starting from the very rudimentary roots of number sense, it proceeds all the way to the modern notion of number. It traces how numbers change from being a tool for counting, quantifying and measuring, to becoming abstract objects. It covers the number systems of all major world civilizations, and the various types of numbers -- natural numbers, negative numbers, rational numbers, irrational numbers, real numbers, complex numbers, algebraic numbers, transcendental numbers and transfinite numbers, along with their history. It also covers special numbers -- zero, prime numbers, Fibonacci numbers, infinity, pi, e, i and the various orders of infinity. It reveals the paradoxical fact that while each of these types or instances of number can be precisely defined, the precise definition of numbers as a whole has kept changing.

Course Purpose: This course aims to develop in students an appreciation for the contributions of various cultures to the growth and development of ideas in mathematics, specifically number systems and number theory. The evolving notion of number shows that mathematics is not just a collection of fixed definitions and formulas to be applied. Intuition, visualization and multiple representations were an important part of the way humans handled numbers, and when understood in historical context, can motivate students towards looking at mathematics as a creative art. Mathematical ideas can also be shown to be very effective in modeling the physical and natural world.

Pre-requisites and Co-Requisites: The students taking this course are expected to have completed Algebra I. Upon the completion of this course, students will be ready to do a course on number theory and certain topics in discrete mathematics. They would also have a unique perspective on the world history of numbers, which can complement a course on AP World History.

Course materials:

Burton, David: *The History of Mathematics: An Introduction*. McGraw-Hill Science/Engineering/Math, 2010
Ifrah, Georges: *The Universal History of Numbers*. John Wiley & Sons, Inc., 2000
Burger, Edward B: *Zero to Infinity: A History of Numbers.* The Teaching Company, 2007
Dunham, William: *Journey through Genius: The Great Theorems of Mathematics.* Penguin Books, 1990.

Course Content:

The course will be divided into the following units:

Numbers in prehistory

What sort of number sense have researchers found in the animal kingdom? How did humans count before the invention of writing? Students will learn about the use of Tally marks, Body parts counting, Knots in cords, Milk sticks, Quipus, etc. They will learn how Neolithic farmers might have keep count of their animals using pebbles, before the invention of writing. A brief introduction to Set Theory and the notion of one-to-one correspondence would be given.

How did pebble counting evolve into a complex system of clay tokens in the Neolithic times? How might an accounting system of clay tokens have further evolved to lead up to the invention of writing in the Sumerian civilization? We will look at the possible origins of the Cuneiform script and the Egyptian hieroglyphs.

Numbers in Ancient Civilizations

The course will then take a tour of the number systems in different civilizations. Students will learn about the decimal system of the Egyptians, which nevertheless lacks place value. After learning about the number system, and various arithmetic operations in it (addition, subtraction, multiplication and division), students will do some sample problems from the Rhind Papyrus. They will learn how the Egyptians used unit fractions.

They will next learn about the sexagesimal Babylonian number system with place value, and how the basic arithmetic operations like addition, subtraction, multiplication and division were done in that number system, along with calculations on fractions.

In a similar way, other number systems will be covered: the vigesimal Mayan number system, the Chinese bamboo or rod numeral decimal system, the Ionic and Attic number systems of the Greeks, and the Roman number system. The number system in India, and its transmission to medieval Europe through the Arabic numerals will be studied.

The development of zero will be traced from its use as a placeholder in Mesopotamia and Mayan civilizations to its treatment as an independent number by Brahmagupta in India.

Due to its importance in computer systems, the Binary Number System and Binary Arithmetic will also be taken up here, starting from Pingala's work on prosody. A brief look at the origin and evolution of mathematical symbols in use today will end this unit.

Number Theory in Ancient Greece

The early development of number theory in ancient Greece will be studied in this unit. Students will learn about Pythagoras and the mystical role of numbers in the Pythagorean religion, along with visual representations of numbers (in the form of triangles, squares, etc) using pebbles. The Pythagorean discovery of numerical ratios underlying the octave and harmony in music and (by extension) nature will be examined. The Pythagorean Theorem and its proof, along with prior evidence of its knowledge in Mesopotamia, Egypt and India will be studied. The Commensurability assumption and the discovery of its violation by the Pythagoreans will be taken up, showing that certain geometrical lengths cannot be represented as rational numbers, leading to questions about the separation of number theory and geometry.

Students will then learn about the Golden Ratio: Can a line segment be divided into two unequal parts such that the ratio of the larger part to the smaller part is the same as the ratio of the whole line segment to the larger part? They will see how to visualise the Golden Ratio in a Golden rectangle.

Prime numbers and the Fundamental Theorem of Arithmetic will be taken up, and students will learn about open problems in number theory, like the Goldbach Conjecture and Twin Prime Conjecture. Euclid's proof that there is an infinite number of primes will be presented, along with the Prime Number Theorem. Euclid's Algorithm for computing the Greatest Common Divisor (GCD) of two numbers will also be studied. The Sieve of Eratosthenes, used for computing all prime numbers up to a certain number will be studied.

Sequences and Series

Efficiently adding up long sequences of numbers will be the focus of this unit. Students will review the triangular numbers of the Pythagoreans, and then look at the sum of the first n natural numbers through the story of Carl Gauss' childhood forays into mathematics. Emphasis will be given on geometric visualization of the sum, and other ways of representing it. Students will be expected to compute the sum of the first n odd (or even) numbers using multiple representations.

These sequences will then be generalized to Arithmetic sequences, and their representation using a pattern of linear growth will be presented. The sum of n terms of an arithmetic sequence (called arithmetic series), calculated using both algebra and geometric visualization, will be presented. Geometric sequences and their representation using a pattern of exponential growth will be presented. The sum of n terms of a geometric sequence (called geometric series), will be calculated, and real-world examples of arithmetic and geometric series will be studied. from biology to finance. Students will learn to calculate the sum of an infinite geometric series (whose terms successively become smaller) using both algebra and geometric visualization.

Telescoping series and the Method of Differences will also be studied.

Numbers in Pre-Modern and Modern times

This unit will begin with a brief history of the Fibonacci number sequence, and the relationship between Fibonacci numbers and the Golden ratio. Examples of Fibonacci sequence of numbers will be drawn from the physical and natural world. The Binet formula for the nth Fibonacci number will be derived. Students will learn about recurrence sequences and how the Fibonacci sequence can be expressed as one. They will be asked to develop recurrence sequences for some additional problems, such as the Tower of Hanoi problem.

Real numbers can be represented as a continuum along a number line, as was done by Simon Stevin. Based on this, students will note how to represent any real number in expanded decimal form, and more generally, in expanded form in any base. Rational numbers have periodic expansions while the expansion of irrational numbers never becomes periodic. Students will be asked to think about whether rational numbers are more in number, or irrational numbers. Comparing their relative densities on the number line will lead to an examination of Borel's notion of normal numbers. The Cartesian coordinate system, and its three dimensional extension, based on the use of multiple number lines perpendicular to each other, will provide the opportunity to take another look at the Pythagorean theorem and how admitting irrational numbers as numbers would preserve the integration of number theory and geometry.

The ideas of Cantor and One-to-one correspondence in set theory will set the stage for a discussion of infinity. The sets of integers and rational numbers will be shown to be countable, i.e, of the same 'size' as the set of natural numbers. The union of any finite number of countable sets will be shown to be countable. The countable union of countable sets will also be shown as countable. And then an insightful 'diagonalization' argument by Cantor will show the set of real numbers as being 'uncountable' or larger in size than the set of natural numbers. This will establish that infinity comes in different sizes. Another diagonalization argument will show that the power set of real numbers is a larger infinite set than the set of real numbers. Questions like "Is there a largest infinity?" and "Is there an infinity between the sizes of natural numbers and real numbers?" will end this section.

Other special types of numbers will be studied. These include:
a) pi : How was the ratio of the circumference and diameter of a circle understood in Egypt, Mesopotamia, India and China? Students will learn about the approximation of pi derived by Archimedes. Modern developments will include the examination of the infinite series of Leibnitz, Gregory and Madhava.
b) e : Starting from the evolution of interest rates in the Greek, Roman and Christian worlds, students will learn how Euler's number e emerged in the context of calculating compound interest. The relationship between natural logarithms and e will also be looked at.
c) i: Students will learn about the imaginary number i, complex numbers and the complex plane.
Numbers that can or can't be solutions to polynomial equations with integer coefficients (Algebraic and Transcendental numbers) will be studied. The unit will end with what is touted

to be the most famous formula in all of mathematics, connecting together different numbers studied in the course:

$$e^{i\pi} + 1 = 0$$

Key assignments:

1. Students will be asked to study the literature to write a short paper on any number system of their choice that is NOT covered in this course. It could be a different number system from one of the cultures we look at, which influenced or was influenced by some number system we cover. Or it could be an independent number system from a different culture. Students will need to explain the types of arithmetic operations performed in that system, in that historical context. They should also compare that number system with any of the number systems we covered, and explain if there were any historical influences of one on the other, and assess their relative efficiency in doing different types of calculations.

2. Students will be asked to take the help of geometric visualization and/or algebra to derive expressions for

a. The sum of the squares of the first n natural numbers
b. The sum of the cubes of the first n natural numbers
c. The sum of the first n terms of an arithmetico-geometric sequence.

Their written work on the same should explain all steps.

3. Students will do a literature survey and write a short paper about different kinds of interest rates prevalent today and why they are different from each other. Students will focus on 3 important rates: The Federal Reserve Rate, LIBOR rates and the US Treasury Yield Rates for different time durations. They will also be solving problems in which they will convert the annual, semi-annual and quarterly rates for the 3 rates to their equivalent continuously compounded rate. This problem solving exercise will also illustrate a convenient property of the continuously compounded rate in that it scales with time unlike discrete compound rate.

4. Students will study the relationship between perfect numbers and Mersenne primes and then work on the proof of the Euler-Euclid theorem as an exercise. The theorem states that an even positive integer is a perfect number, that is, equals the sum of its proper divisors, if and only if it has the form $2^{p-1}M_p$ where M_p is a Mersenne prime. Students would be expected to work in one direction independently (all Mersenne primes of the form $2^{p-1}M_p$ are perfect numbers) and should be able to work on the converse with some assistance.

5. Students will write a short essay about Georg Cantor and his life. They will focus on his proof showing that real numbers are more numerous than rational numbers and his definition of the concept of transfinite number. In addition, students will also trace reasons for which great mathematicians like Poincare opposed Cantor and how Cantor's theory was finally accepted by the mathematicians. This essay will expose students to the historical development of new theories and more importantly how theories that are opposed fiercely at one time might finally be accepted by the mathematical community at large.

The course content will be implemented through a combination of lectures, in-class problem solving (on the mathematics and science portions), in-class discussions, weekly problem sets given as homework, key assignments and projects such as the above, and written exams.